Exact coefficients of partition functions via stability

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Partition functions

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- Arise in combinatorics as graph polynomials.
- Main example: matchings in regular graphs.
- Other examples: independent sets and colourings in regular graphs, triangle-free graphs, etc.

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- The same idea can be used for independent sets, colourings, etc.

Properties of the partition function

$$Z_G(\lambda) = \sum_{M} \lambda^{|M|} = \sum_{k \geq 0} m_k(G) \lambda^k$$

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- The average size of a matching $\mathbf{M}$ from the monomer-dimer model is

$$\mathbb{E}|\mathbf{M}| = \frac{\sum_M |M| \lambda^{|M|}}{Z_G(\lambda)} = \frac{\lambda Z_G'(\lambda)}{Z_G(\lambda)} = \lambda \frac{\partial}{\partial \lambda} \log Z_G(\lambda)\,.$$

Previous work

Consider the family of d-regular graphs and let $H_{d,n}$ be the disjoint union of $n/2d$ copies of $K_{d,d}$.

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- In previous work we showed that for all $\lambda > 0$, $H_{d,n}$ maximises the partition function over n-vertex, d-regular graphs.
- In fact, we showed that $H_{d,n}$ maximises

$$\frac{1}{|E(G)|}\mathbb{E}|\mathbf{M}| = \frac{\lambda}{|E(G)|}\frac{\partial}{\partial\lambda}\log Z_G(\lambda)$$

  over all d-regular graphs.

Main results

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2. If G contains no copy of $K_{d,d}$, should $Z_G(\lambda)$ be significantly smaller than $Z_{H_{d,n}}(\lambda)$? This is a question of stability.

We prove in a general way that a strong form of 2 holds, and that from such a result, 1 follows for a wide range of parameters.

Stability

In our previous work we showed that for $\lambda > 0$ and d-regular G,

$$\frac{1}{|V(G)|} \log Z_G(\lambda) \leq \frac{1}{2d} \log Z_{K_{d,d}}(\lambda) ,$$

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Theorem

Let G be a d-regular graph which contains no copy of $K_{d,d}$. Then there exists a continuous function $s(d, \lambda)$ which is strictly increasing in λ, and satisfies $s(d, 0) = 0$, such that the following holds for $\lambda \geq 0$,

$$\frac{1}{|V(G)|} \log Z_G(\lambda) \leq \frac{1}{2d} \log Z_{K_{d,d}}(\lambda) - s(d, \lambda).$$

Proof: linear programming with local constraints

- Let $\alpha_G(\lambda) = \frac{1}{|E(G)|}\mathbb{E}|\mathbf{M}| = \frac{\lambda}{|E(G)|}\frac{\partial}{\partial\lambda}\log Z_G(\lambda)$.

Proof: linear programming with local constraints

- Let $\alpha_G(\lambda) = \frac{1}{|E(G)|}\mathbb{E}|\mathbf{M}| = \frac{\lambda}{|E(G)|}\frac{\partial}{\partial\lambda}\log Z_G(\lambda)$.
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- The maximum value of $\alpha_G(\lambda)$ over all d-regular graphs can be expressed as a linear program which depends only on d, λ.
- The constraint that G contains no copy of $K_{d,d}$ can be naturally added to the program, yielding:

Lemma

For any d-regular G which contains no copy of $K_{d,d}$,

$$\alpha_G(\lambda) \leq \alpha_{K_{d,d}}(\lambda) - c(d,\lambda)\,.$$

Proof: simple calculus gives the stability result

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Let G contain no $K_{d,d}$. Then

$$\frac{1}{|V(G)|}\log Z_G(\lambda) = \frac{d}{2}\int_0^\lambda \frac{\alpha_G(t)}{t}\,\mathrm{d}t$$

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Let G contain no $K_{d,d}$. Then

$$\begin{aligned}
\frac{1}{|V(G)|}\log Z_G(\lambda) &= \frac{d}{2}\int_0^\lambda \frac{\alpha_G(t)}{t}\,\mathrm{d}t \\
&\le \frac{d}{2}\int_0^\lambda \frac{\alpha_{K_{d,d}}(t) - c(d,t)}{t}\,\mathrm{d}t \\
&= \frac{1}{2d}\log Z_{K_{d,d}}(\lambda) - \underbrace{\frac{d}{2}\int_0^\lambda \frac{c(d,t)}{t}\,\mathrm{d}t}_{s(d,\lambda)}
\end{aligned}$$

Exact bounds on coefficients for almost all sizes

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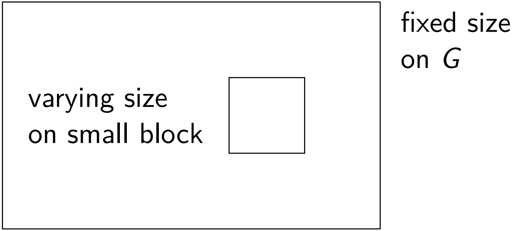

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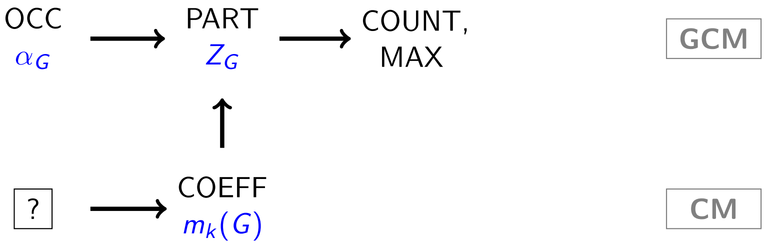
  - value of each coefficient: COEFF — now almost solved

The big picture



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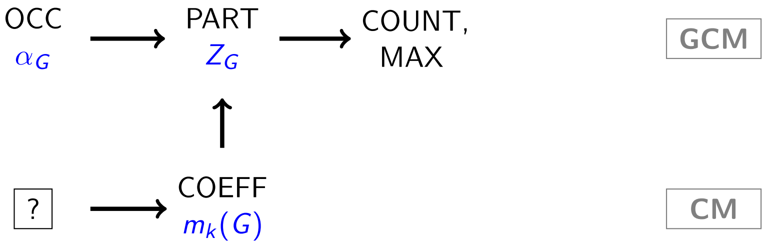


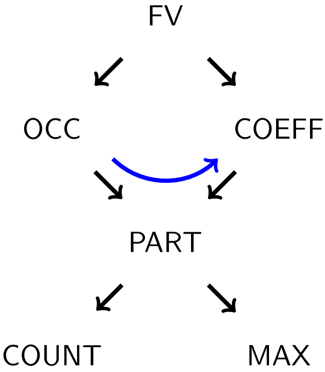
The missing piece is the free volume:

$$f_G(M) = \text{set of edges which could be added to } M\,,$$

$$F_{G,k}(\lambda) = \mathbb{E}|f_G(\mathbf{M}_k)| = (k+1)\frac{m_{k+1}(G)}{m_k(G)}\,,$$

where $\mathbf{M}_k$ is a uniformly random matching of size k in G.

Another big picture



We conjecture that $H_{d,n}$ maximises the free volume for all k, i.e. has property FV.