Bounding the (list) chromatic number of triangle-free graphs

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Basic graph theory concepts

- Δ is **maximum degree**, α is the size of the largest **independent set**, χ is the **chromatic number**
- ρ is the **Hall ratio** $\max_{\emptyset \neq H \subset G} \frac{|H|}{\alpha(H)}$
- χ_f is the least k such that there's a probability distribution on independent sets such that for all v, $\Pr(v \in I) \geqslant 1/k$
- χ_ℓ is the least k such that whenever the vertices of a graph are given lists of k allowed colors, there is a proper coloring using allowed colors
- χ_c is more general: for each edge uv specify an arbitrary matching of 'forbidden pairs' from the lists $L(u)$ and $L(v)$
- In any graph $\rho \leqslant \chi_f \leqslant \chi \leqslant \chi_\ell \leqslant \chi_c$

How does local structure constrain $\chi(G)$?

- Greedy algorithm: $\chi \leqslant \Delta + 1$
- Brooks (1941): this is tight only for odd cycles and cliques
- Descartes (Tutte, 1954): there are triangle-free graphs with arbitrarily large chromatic number
- Vizing (1968): posed the problem of bounding χ in terms of Δ for **triangle-free graphs**
- Various authors (1977-8): for triangle-free graphs $\chi \leqslant \frac{3}{4}(\Delta + 2)$;
  Kostochka (1978): $\chi \leqslant \frac{2}{3}(\Delta + 3)$
- Johansson (1996): $\chi \leqslant O(\Delta / \log \Delta)$
  Molloy (2019): $\chi \leqslant (1 + o(1))\Delta / \log \Delta$

Refined notions of coloring

List coloring

- Greedy algorithm still works: $\chi_\ell \leqslant \Delta + 1$
- Vizing (1976) gave the list version of Brooks' theorem
- Methods of Johansson (1996) and Molloy (2019) also apply to list coloring: for triangle-free graphs $\chi_\ell(G) \leqslant (1 + o(1))\Delta/\log\Delta$

Correspondence coloring

- Greedy algorithm *still* works: $\chi_c \leqslant \Delta + 1$
- Bernshteyn, Kostochka and Pron (2017) gave a corresponding version of Brooks' theorem
- Bernshteyn (2016, 2019) adapted the methods of Johansson and Molloy: for triangle-free graphs $\chi_c \leqslant (1 + o(1))\Delta/\log\Delta$

The χ-Ramsey problem for triangle-free graphs

- Erdős (1967) asked for the greatest chromatic number among n-vertex triangle-free graphs
- Related to the classic Ramsey problem of finding the largest independent set in triangle-free graphs as iteratively pulling out such sets gives a coloring, cf. Erdős and Hajnal (1985)
- Each of $\rho \leqslant \chi_f \leqslant \chi \leqslant \chi_\ell \leqslant \chi_c$ has a Ramsey-type question
- Bounding $\rho = \max_{\emptyset \neq H \subset G} \frac{|H|}{\alpha(H)}$ in K_r-free graphs is classic Ramsey theory, bounding each of the others is **harder**
- Mostly we don't know the correct dependence of ρ on n, but the question of how close the bound for χ can be made to the best-known for ρ is still interesting

The χ-Ramsey problem for triangle-free graphs

- Ajtai, Komlós and Szemerédi (1980): $\rho \leqslant O(\sqrt{n/\log n})$
- Shearer (1983) improved to $(\sqrt{2}+o(1))\sqrt{n/\log n}$
- Iterating gives $\chi \leqslant (2\sqrt{2}+o(1))\sqrt{n/\log n}$
- What is that extra factor '2' doing there?
- Pulling out independent sets does not seem to work for list (or correspondence) coloring. What is the correct order of growth?
- Cames van Batenburg, de Joannis de Verclos, Kang, and Pirot (2020) asked such questions, while proving
  $\chi_f \leqslant (2+o(1))\sqrt{n/\log n}$ and $\chi_\ell \leqslant O(\sqrt{n})$
- The correct growth rate for χ_c is actually $\Theta(n/\log n)$
  (cf. Král', Pangrác, and Voss 2005 and Bernshteyn 2016, 2019)

Our results in triangle-free graphs

- For chromatic number we match the previous bound for fractional chromatic number: $\chi \leqslant (2+o(1))\sqrt{n/\log n}$
- So we now ask why there is an extra factor '$\sqrt{2}$'...
- For list chromatic number we show $\chi_\ell \leqslant O(\sqrt{n/\log n})$
- Our method highlights subtle aspects of list coloring: bounds in terms of **color-degree**, etc...
- Adapting existing methods also yields bounds in terms of the number of **edges** or **genus** that are tight up to a constant factor, cf. Poljak and Tuza (1994), Nilli (Alon, 2000), Gimbel and Thomassen (2000)
- The paper is here: `https://doi.org/10.1137/21M1437573` and here: `https://arxiv.org/pdf/2107.12288`

The proof sketch for chromatic number

- Ignore all $o(1)$ terms and prove $\chi \leqslant 2\sqrt{n/\log n}$ by induction
- We are done by Molloy's theorem if $\Delta \leqslant \sqrt{n \log n}$
- Let v be a vertex of larger degree and let $G' = G - N(v)$ on $n' \leqslant n - \sqrt{n \log n}$ vertices
- Since $N(v)$ is independent, $\chi(G) \leqslant 1 + \chi(G')$ and by induction,

$$\chi(G') \leqslant 2\sqrt{\frac{n'}{\log n'}} \lesssim 2\sqrt{\frac{n - \sqrt{n \log n}}{\log n}} \leqslant 2\sqrt{\frac{n}{\log n}} - 1$$

- **Exercise:** extend this sketch to a correct proof!

The idea for list chromatic number

- Let each vertex have a list $L(v)$ of allowed colors
- Can't assume all neighbors of v are allowed the same color
- But there's a notion of **color-degree** that works: for $c \in L(v)$, $\deg_L(v, c)$ is the number of neighbors of v that have color c on their list
- Happily, due to Amini and Reed (2008), Alon and Assadi (2020), or even Anderson, Bernshteyn and Dhawan (2022+) we have a color-degree analogue of Johansson's theorem

Theorem

If L is a list-assignment for a triangle-free graph G such that $|L(v)| \geqslant (4 + o(1))d/\log d$ and every $\deg_L(v, c) \leqslant d$, then G admits an L-coloring

The proof for list chromatic number

- If all color-degrees are $O(\sqrt{n \log n})$ then done by theorem
- Otherwise, there's a vertex v, a color $c \in L(v)$, and a large set $S_c \subset N(v)$ such that $c \in L(w)$ for $w \in S_c$
- Color S_c with c and let $G' = G - S_c$ with lists $L'(w) = L(w) \setminus \{c\}$
- Observe that G admits an L-coloring if G' admits an L'-coloring
- Set up the constant factors and an induction hypothesis such that G' admits an L'-coloring by induction
- Some constant factor loss due to pesky '4' in the theorem
- Improving '4' to '1' in the theorem is an open problem, conjectured by Cambie and Kang (2021) and Anderson, Bernshteyn and Dhawan (2022+); would imply the same bound on χ_ℓ that we proved for χ

Further reducing the constant

- Kelly and Postle (2018+) posed a conjecture which would allow us to remove the '$\sqrt{2}$' and match Shearer's upper bound for ρ in triangle-free graphs with a bound for χ_f
- Their conjecture is equivalent to the existence of a probability distribution on independent sets in a triangle-free graph such that for every vertex v

$$\Pr(v \in I) \geqslant (1 - o(1)) \frac{\log \deg(v)}{\deg(v)}$$

cf. Shearer (1991) $\alpha \geqslant \sum_{v \in V(G)} (1 - o(1)) \frac{\log \deg(v)}{\deg(v)}$

- I do not know of any analogous conjecture/argument that would show we can remove a '$\sqrt{2}$' for χ or χ_ℓ
- This seems interesting!

Further technicalities I

- The theorem of Anderson, Bernshteyn and Dhawan actually states that for an **arbitrary** graph G with list assignment L, if
  (a) $|L(v)| \geqslant (4 + o(1))d/\log d$
  (b) $\deg_L(v, c) \leqslant d$ for all color degrees
  (c) for all colors c, the subgraph of G induced by the vertices with c on their lists is triangle-free

  then G admits an L-coloring.
- That is, we can push the triangle-freeness onto the 'cover graph' which represents conflicts between colors on lists of G
- Actually, their theorem holds for correspondence coloring...
- Still, it seems reasonable that '4' can be reduced to '1'

Further technicalities II

- An alternative perspective seeks results with 'local' bounds: Let G be a triangle-free graph with list assignment L such that for all vertices v and colors $c \in L(v)$ we have $|L(v)| \geqslant (1+\varepsilon) \deg_L(v,c) / \log \deg_L(v,c)$
- Additional conditions are needed for an L-coloring (D., de Joannis de Verclos, Kang, and Pirot 2020) e.g. for some $d \geqslant d_0(\varepsilon)$ we need $\text{polylog}(d) \leqslant \deg_L(v,c) \leqslant d$
- Kelly (2019) conjectures (roughly) that in this case G should indeed admit an L-coloring
- D., de Joannis de Verclos, Kang, and Pirot (2020) proved this when the bound is in terms of $\deg(v)$ instead of color-degree
- Kelly showed that the full version of his conjecture implies the probability distribution conjecture of Kelly and Postle!
- What about pushing triangle-freeness into the cover?

Further technicalities III

- Versions of Molloy’s theorem are known for other ‘locally sparse’ conditions which invites applications of our methods to a range of χ-Ramsey questions
- We decided not to do this, as it largely concerns chasing constant factors in bounds we don’t know are tight
- Many of the best-known bounds here follow from adaptations of Molloy’s method (D., Kang, Pirot and Sereni 2020+) but these methods are not known to work with color-degrees
- Recent works of Anderson, Bernshteyn and Dhawan (2021+, 2022+) are based on Johansson’s earlier approach and give color-degree results in $K_{t,t}$ or $K_{1,t,t}$-free graphs

Final conjecture

The ideal result is that L-colorings exist when for all $c \in L(v)$, $|L(v)| \geqslant (1 + o(1)) \deg_L(c, v) / \log \deg_L(c, v)$ and the cover is triangle-free, with the mildest lower bounds on degrees possible (in fact, the correspondence version of this)

Thank you