TIGHT BOUNDS ON THE COEFFICIENTS OF PARTITION FUNCTIONS VIA STABILITY (EXTENDED ABSTRACT)

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Abstract. We show how to use the recently-developed occupancy method and stability results that follow easily from the method to obtain extremal bounds on the individual coefficients of the partition functions over various classes of bounded degree graphs.

As applications, we prove new bounds on the number of independent sets and matchings of a given size in regular graphs. For large enough graphs and almost all sizes, the bounds are tight and confirm the Upper Matching Conjecture of Friedland, Krop, and Markström, and a conjecture of Kahn on independent sets for a wide range of parameters. Additionally we prove tight bounds on the number of q-colorings of cubic graphs with a given number of monochromatic edges, and tight bounds on the number of independent sets of a given size in cubic graphs of girth at least 5.

1. Introduction

A Gibbs measure is a probability distribution on states of a system in which each state possesses an energy, and a state occurs with probability proportional to an exponential in its energy. The normalising constant, called the *partition function*, has a wide range of important properties. Of particular interest here are the facts that expectations over the Gibbs measure can be computed from the partition function, and that the coefficients of the partition function correspond to the numbers of states with fixed energy.

Independent sets, matchings, and colourings are fundamental objects of study in graph theory. The fact that a many different phenomena can be represented in graphs means that bounds on the number of these objects appear throughout mathematics; such as a question of Granville on which d-regular graph on n vertices has the most independent sets (see [1]). By considering appropriate Gibbs measures, some of these questions can be framed in terms of bounds on the partition function of models from statistical physics. The hard-core model and monomer-dimer model correspond to independent sets and matchings respectively. We also consider the Potts model and how it relates to colourings of graphs.

Stability is a wide-ranging theme in combinatorics which concerns whether structures that nearly attain an extreme value of some parameter must closely resemble one another. The archetypal stability result in graph theory is the theorem of Erdős and Simonovits [10, 15] which states that K_{r+1}-free graphs with close to the maximum possible number of edges must closely resemble a complete, balanced r-partite graph.

In this extended abstract we discuss how stability results follow easily from the recent occupancy method [6], which gives bounds on the partition function in a

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variety of Gibbs measures over certain classes of graphs [4, 6, 7, 13]. As an application we derive bounds on the individual coefficients of the partition functions. In particular, for large enough graphs and almost all sizes, we give tight upper bounds on the number of independent sets and matchings of fixed size in regular graphs, confirming conjectures of Kahn [12] and Friedland, Krop, and Markström [11] for a wide range of parameters. We give details in a forthcoming paper [8], here choosing to focus on the example of matchings in regular graphs.

2. Stability in extremal results on matchings

Bregman's theorem [3] implies that when $2d$ divides n, the balanced bipartite graph on n vertices of average degree d with the most perfect matchings is a disjoint union of $K_{d,d}$'s, denoted $H_{d,n}$. Kahn and Lovász removed the bipartite restriction in an unpublished paper, but various proofs of the result now exist (e.g. [2]). The entropy method has been a powerful tool in the area, including the proofs of Bregman's theorem and the Kahn–Lovász extension by Radhakrishnan [14], and Cutler–Radcliffe [5] respectively. In what follows we restrict our attention to d-regular graphs.

It is natural to state recent developments of these results in the language of statistical physics. The *monomer-dimer model* on a graph G at fugacity λ is the probability distribution on matchings M in G given by

$$\Pr[M] = \frac{\lambda^{|M|}}{Z_G^{\text{match}}(\lambda)} ,$$

where the denominator is the *partition function* (also known as the matching polynomial or matching generating function) $Z_G^{\text{match}}(\lambda) = \sum \lambda^{|M|}$ where the sum is over the set $\mathcal{M}(G)$ of matchings in G. We may interpret the Kahn–Lovász result as a bound the coefficient of $\lambda^{n/2}$ in $Z_G^{\text{match}}(\lambda)$.

Continuing with the language of statistical physics, we define the *occupancy fraction* as the expected fraction of edges in a random matching $\mathbf{M}$ drawn from the monomer-dimer model (at fugacity λ),

$$\alpha_G^{\text{match}}(\lambda) = \frac{2}{dn}\mathbb{E}_{G,\lambda}|\mathbf{M}| ,$$

and note the following relation to the partition function,

$$\begin{aligned}\alpha_G^{\text{match}}(\lambda) = \frac{2}{dn}\mathbb{E}_{G,\lambda}|\mathbf{M}| &= \frac{2}{dn}\frac{\sum_{M\in\mathcal{M}(G)}|M|\lambda^{|M|}}{Z_G^{\text{match}}(\lambda)} \\ &= \frac{2}{dn}\frac{\lambda}{Z_G^{\text{match}}(\lambda)}\frac{\partial}{\partial\lambda}Z_G^{\text{match}}(\lambda) \\ &= \frac{2\lambda}{dn}\frac{\partial}{\partial\lambda}\log Z_G^{\text{ind}}(\lambda) .\end{aligned}$$

In [6] the authors proved that $K_{d,d}$ maximises the occupancy fraction $\alpha_G^{\text{match}}(\lambda)$ over d-regular graphs, using an *occupancy method* that seems to differ substantially from the entropy method. That $H_{d,n}$ maximises $Z_G^{\text{match}}(\lambda)$ over n-vertex d-regular graphs follows from the above calculation, and in the limit $\lambda \to \infty$ we recover maximality of the number of perfect matchings.

Given these results, the natural form of stability to consider is whether a d-regular graph that is far from $H_{d,n}$ in some sense must contain many fewer matchings. A strong form of this stability holds for graphs which do not contain a copy of $K_{d,d}$; they have exponentially fewer matchings.

Theorem 1 (Davies, Jenssen, Perkins, Roberts [8]). *Let G be an n-vertex, d-regular graph which contains no copy of $K_{d,d}$. Then there exists a continuous function*

$s(d,\lambda)$ which is strictly increasing in λ, and satisfies $s(d,0)=0$, such that the following holds for $\lambda \geq 0$,

$$Z_G^{\mathrm{match}}(\lambda) \leq e^{-s(d,\lambda)n} Z_{H_{d,n}}^{\mathrm{match}}(\lambda)\,.$$

For the hard-core model a similar stability result for $\lambda = 1$ was given in [9], and with some work it should be possible to deduce stability in Bregman's theorem from Radhakrishnan's proof [14]. The main benefit of obtaining stability via the occupancy fraction is generality, but in applications [8] we make crucial use of the monotonicity of $s(d,\lambda)$ in λ. The methods of [6] have been successfully applied to analogous problems on independent sets and colourings [4, 7, 13]; in each of these cases finding a unique connected extremal graph for the problem of maximising a derivative of the free energy of a probabilistic model. Our general method for proving stability applies in each case, yielding a result of the same form, see [8].

3. Tight bounds on coefficients of the partition function

We apply stability results of the form of Theorem 1 to prove tight bounds on the coefficients of the relevant partition functions for a wide range of parameters. Let $m_k(G)$ be the number of matchings with k edges in G.

Theorem 2 (Davies, Jenssen, Perkins, Roberts [8])**.** *For all $d \geq 2$ and $\epsilon > 0$ there exists $N = N(d,\epsilon)$ such that the following holds. Suppose that $n \geq N$ and n is divisible by $2d$. Let G be any d-regular graph on n vertices. Then for all $k > \epsilon n$, $m_k(G) \leq m_k(H_{d,n})$.*

This solves the Upper Matching Conjecture of Friedland, Krop, and Markström [11] for the range of k and n given. We give the details in [8], including applications of the method to independent sets and q-colorings.

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